

OTS: 60-31-179

JPRS: 3290

24 May 1960

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ESTIMATE OF USEFUL SIGNAL

- USSR -

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19990322 079

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USING LINEAR THEORY FOR OBTAINING UNBIASED
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Following is a translation of the article "Unbiased Estimate of a Useful Signal Which Depends Non-Linearly on Unknown Parameters" by I. A. Boguslavskiy in Avtomatika i Telemekhanika, Vol. XXI, No. 1, Moscow, January 1960, pages 42-47.

A method is presented which makes it possible in a number of cases to employ linear theory for obtaining, in the presence of noise, unbiased estimate of a useful signal which depends nonlinearly on unknown parameters.

* * * * *

References /1 - 4/ have solved the problem of obtaining, in the presence of noise, the optimum estimate from some viewpoint of a useful signal which is a linear function of unknown parameters. In a number of cases of practical importance the useful signal is a non-linear function of these parameters. The general solution of the corresponding non-linear problem was obtained by V. S. PUGACHEV.

(Note: Reports of 16 February and 2 March 1959 at the Seminar on Probabilistic methods of the Theory of Automatic Control at the Institute of Automatics and Telemechanics, Academy of Sciences USSR.)

However, there is a definite practical interest in artificial methods which effectively solve this problem in some cases through the use of linear methods of statistical dynamics. One of these methods is set forth below. The method makes it possible, for example, to accomplish rather simply the unbiased filtration of a random process whose useful signal satisfies a non-linear differential equation of a definite type with unknown initial conditions. The method can be used for smoothing, without dynamic errors, the coordinates and projections of the velocity of an artificial earth satellite cosmic rocket traveling on the passive sector of its motion, etc.

Description of the Method

There is observed a random process $Z(t)$ in the form

$Z(t) = \Phi(c_1, c_2, \dots, c_n, t) + m(t),$

where Φ is the useful signal, c_h random parameters, and $m(t)$ noise.

Let it be possible to denote a transformation I , which converts Φ into the function of the form $\sum_{k=1}^N c_k W_k(t)$ with the a priori known $W_k(t)$ and transformation II , which converts the result of the employment of transformation I back into the function Φ . In addition, let us assume that the random process $Z_1(t)$ is the result of application of transformation I to $Z(t)$ can be represented approximately in the form

$$Z_1(t) = \sum_{k=1}^N c_k W_k(t) + m_1(t),$$

where $m_1(t)$ is a random noise process whose correlation function can be expressed through the probabilistic characteristics of the noise $m(t)$.

Then the scheme for obtaining an unbiased estimate of the useful signal can be built up as follows:

1. Apply transformation I to $Z(t)$.
2. Send $Z_1(t)$ to the input of the optimum unbiased filtration link which is constructed by known linear methods on the basis of the data on functions $W_k(t)$ and the noise correlation function $m_1(t)$; in the memory time T selected at the output of the link is developed the function $\sum_{k=1}^N c_k W_k(t)$ with decreased random errors.
3. Apply transformation II to the output of the link.

Realization of the described sequence of operations insures the construction in the time T of a function $\Phi(c_1, c_2, \dots, c_n, t)$ without dynamic errors and with reduced random errors, if transformation II "raises" the level of noise at the output of the optimum linear link only insignificantly.

Let it be known that the useful signal Φ satisfies a non-linear differential equation of the form

$$a_0(t) \frac{d^n \Phi}{dt^n} + a_1(t) \frac{d^{n-1} \Phi}{dt^{n-1}} + \dots + a_n(t) \Phi = F(t, \Phi) \quad (1)$$

with unknown initial conditions

$$c_1 = \phi(0), c_2 = \dot{\phi}(0), \dots, c_n = \phi^{(n-1)}(0).$$

Due to the presence in the right-hand member of (1) of the function $F(t, \underline{\phi})$ in this case, the useful signal is a non-linear function of the unknown parameters c_1, c_2, \dots, c_n . From (1) it follows that

$$\underline{\phi}(c_1, c_2, \dots, c_n, t) = \sum_{k=1}^n c_k W_k(t) + \int_0^t W(t, \tau) F[\tau, \underline{\phi}] d\tau, \quad (2)$$

where $W_k(t)$ is the solution of the homogeneous equation

$$a_n(t) \frac{d^n W_k}{dt^n} + a_{n-1}(t) \frac{d^{n-1} W_k}{dt^{n-1}} + \dots + a_1(t) \frac{d W_k}{dt} + a_0(t) W_k = 0$$

for the initial conditions

$$W_k^{(m)}(0) = \begin{cases} 0 & \text{for } m \neq k, \\ 1 & \text{for } m = k, \end{cases}$$

while $W(t, \tau)$ is the pulse transfer function of the corresponding linear dynamic link

From (2) it is evident that transformation I should be defined by the relation

$$Z_1(t) = Z(t) - \int_0^t W(t, \tau) F[\tau, Z(\tau)] d\tau \quad (3)$$

and transformation II by the relation

$$Z_2(t) = Z^*(t) + \int_0^t W(t, \tau) F[\tau, Z(\tau)] d\tau, \quad (4)$$

where $Z^*(t)$ signifies the result of optimum unbiased filtration of the process $Z_1(t)$. In the time T , given in the solution of the problem of optimization, $Z_2(t)$ will differ from $\underline{\phi}(c_1, c_2, \dots, c_n, t)$ only by random errors.

As a rule, it is possible approximately to set $W_{n-1}(t) \approx W(t)$, if only the link with the pulse transfer function $W(t, \tau)$ possesses substantial filtering properties, and $F[t, \underline{\phi}]$ is of the same order as $\underline{\phi}$. In a manner analogous to the above we can obtain an unbiased estimate of the result of application of any given linear operator S to function $\underline{\phi}$.

In this case

$$Z_2(t) = Z^*(t) + S \left\{ \int_0^t W(t, \tau) F[\tau, Z(\tau)] d\tau \right\},$$

where $Z^*(t)$ is the optimum unbiased estimate in time T of the function $S \left\{ \sum_{k=1}^n c_k W_k(t) \right\}$.

We note that for a given function $\Phi(c_1, c_2, \dots, c_n, t)$ the corresponding non-linear differential equation may be found by exclusion of the parameters c_k from the following system of $n+1$ algebraic equations:

$$\begin{aligned}\Phi &= \Phi(c_1, c_2, \dots, c_n, t), \\ \frac{d\Phi}{dt} &= \Phi'(c_1, c_2, \dots, c_n, t), \\ \frac{d^2\Phi}{dt^2} &= \Phi''(c_1, c_2, \dots, c_n, t).\end{aligned}$$

However the transformation (3) and (4) can be used directly for obtaining an unbiased estimate only in the case when the non-linear member of the obtained differential equation depends only on Φ and t .

Using the concept of Jacobian functions Φ , Φ' , ---- in relation to c_1, c_2 , ---- we can obtain an analytical expression for the condition mentioned.

For $n = 1$ this condition is always satisfied. Thus if, for example,

then

$$\Phi = \frac{c}{c+t},$$

$$\frac{d\Phi}{dt} = -\frac{c}{(c+t)^2} \text{ and } t \frac{d\Phi}{dt} + \Phi = \Phi^3.$$

Supplementing the method described with optimum determination of the derivative functions of $Z_1(t)$ makes it possible also, in conjunction with the method of successive approximations, to find an unbiased estimate when there is a dependence of the non-linear member on the derivative functions of Φ .

To obtain an unbiased estimate by the scheme described does not require the solution of a non-linear differential equation. Therefore, in the example considered below the scheme makes possible an uncomplicated solution of a concrete engineering problem.

Determination of Smoothed Values of Trajectory Elements of an Artificial Earth Satellite

Let us set up a procedure for determining without dynamic errors the smoothed values of the coordinates and projections of the velocities of artificial earth satellites. To shorten the treatment, we will examine the plane case; calculations involved in smoothing non-planar motion of a satellite cause no difficulties.

Let x, y be the coordinates of the satellite relative to a rectangular system of coordinates which do not participate in the Earth's rotation and with its origin at the center of the Earth. If the influence of the braking force of the Earth's atmosphere on the satellite and the nonsphericity of the Earth are not taken into account, x, y satisfy the non-linear differential equations

$$\ddot{x} = -\frac{x}{R^3} (x^2 + y^2)^{3/2}, \quad (5)$$

$$\ddot{y} = -\frac{y}{R^3} (x^2 + y^2)^{3/2}, \quad (6)$$

where R is the radius of the Earth, $g = 9.81 \text{ m/sec}^2$.

Therefore, for example, $x(t)$ is described by the formula

$$x(t) = x(t_0) + \dot{x}(t_0)(t - t_0) - \int_{t_0}^t \frac{1}{R^3} \frac{x(\tau)}{(x^2(\tau) + y^2(\tau))^{3/2}} d\tau \quad (7)$$

or the formula

$$x(t) = x(t_0) \cos \omega t + \frac{\dot{x}(t_0)}{\omega} \sin \omega t - \frac{1}{\omega} \int_{t_0}^t \sin \omega(t - \tau) \frac{x(\tau)}{[x^2(\tau) + y^2(\tau)]^{3/2}} d\tau, \quad (8)$$

where $\omega = \sqrt{\frac{g}{R}}$.

Formula (8) separates the part of the useful signal which depends linearly on the unknown parameters $x(t_0)$ and $\dot{x}(t_0)$ more completely than formula (7). However, the filtering effectiveness in using (7) or (8) is practically identical. Therefore we can employ formula (7), which makes for a simpler form of the linear operator which accomplishes the optimum unbiased filtration of the result of transformation I . Let there be observed the random process

$$z_x(t) = x(t) + n_x(t),$$

where $n_x(t)$ is the random errors in measurement of coordinate $x(t)$ by ground radio technical or optical means.

From (7) it follows that transformation I is defined by the relation

$$I_{x,1}(t) = z_x(t) - \int_{t_0}^t \frac{1}{R^3} \frac{z_x(\tau)}{[z_x^2(\tau) + z_y^2(\tau)]^{3/2}} d\tau, \quad (9)$$

and transformation Π by the relation

$$Z_{x2}(t) = Z_x^*(t) - \int_0^T \frac{gR^2(t-\tau)Z_x(\tau)}{[Z_x^2(\tau) + Z_y^2(\tau)]^{3/2}} d\tau. \quad (10)$$

The specifics of the problem under consideration reduce to the fact that in this case the function $U(t, \tau)$ has no filtering properties (when using (7) or (8) $U(t, \tau)$ is equal respectively to $t - \tau$ or $\frac{1}{\omega} \sin \omega(t - \tau)$). Therefore the dispersion of noise $\eta_1(t)$ contained in $Z_{x1}(t)$ increases infinitely in the course of time.

However, if T is limited to some value completely acceptable for practical purposes (see appendix), then $m_{x1}(t)$ in practice coincides with the noise $n_x(t)$. Therefore for the known correlation function of the noise $n_x(t)$ it is easy to construct a circuit, converting $Z_{x1}(t)$ into $Z_x^*(t)$, for optimum unbiased filtration of the random process whose useful part is a linear function of time.

Transformation Π in the case of limitation of T mentioned above in practice does not reduce the effectiveness of this filtration.

The random errors arising in measurements of the y coordinates are subjected to unbiased filtration in an analogous manner. If it is necessary to find $V_x(t)$, $V_y(t)$ --- the projections of the velocity vector of the satellite --- then transformation I , as previously, is described by relation (9), while transformation Π takes the form:

$$V_x(t) = V_x^*(t) - \int_0^T \frac{gR^2 Z_x(\tau)}{[Z_x^2(\tau) + Z_y^2(\tau)]^{3/2}} d\tau,$$

where $V_x^*(t)$ is the result of optimum unbiased differentiation of the random process $Z_{x1}(t)$.

When the non-sphericity of the Earth is taken into account, the smoothing scheme remains unchanged, if only the constant g in the formulas is replaced by a function of x and y , defined by known relations of potential theory. When the influence of the atmospheric braking force on the motion of the satellite is taken into account, components depending on both x, y and V_x, V_y , are added to the right-hand member of equations (5), (6), and therefore, to the functions under the integral in (9), (10). The form of these components is known if the density of the atmosphere is known as a function of x, y and the coefficient of any resistance as a function of V . Here the procedure becomes somewhat more complex, requires the use of successive approximations, and can be designated in the following form. First V_x and V_y with some dynamic errors neglecting the influence of the braking force are found by the method described above. The V_x and V_y found are used

to construct components which take into account this force and supplements to the functions under the integral in (9) and (10) when the process of determining new, refined values of \dot{x}, \dot{y} is repeated. Carrying out the described procedure several times, we obtain \dot{x} and \dot{y} practically without dynamic errors and with random errors determined, with the limitation of τ , fundamentally by random errors in optimum differentiation. We note that, frequently, successive approximations cannot be employed and that, instead of \dot{x}, \dot{y} , we use the results of numerical differentiation with a specially selected spacing of observed values Z_x, Z_y .

Without alteration, the procedure can also be used for obtaining smoothed values of the coordinates and projections of the velocity of cosmic rockets which are in complex motion under the influence of the gravitation of the Earth and several heavenly bodies. In this case we introduce terms depending on both x, y and the current t , into the right hand members of equations (5) and (6), which take into account the variation of the coordinates of these bodies relative to the Earth during the smoothing process.

APPENDIX

We will designate by $\Delta(t)$ a random constituent contained in the integral components of formulas (9), (10), resulting from the presence of random errors $m_x(t)$, $m_y(t)$ and will designate its dispersion as $\sigma_\Delta^2(t)$.

We will consider that in practice the magnitude of $X^2 + Y^2$ is several orders greater than the magnitude of $Z(x, m_x, y, m_y)$. Therefore

$$\Delta(t) = \int_0^t \frac{X^2(\tau) + Y^2(\tau)}{[X^2(\tau) + Y^2(\tau)]^{3/2}} d\tau$$

and

$$\sigma_\Delta^2(t) \leq \int_0^t \int_0^t (t-\tau_1)(t-\tau_2) q_x(\tau_1, \tau_2) d\tau_1 d\tau_2 \quad (11)$$

where $q_x(t_1, t_2)$ is the correlation function of the random error $m_x(t)$.

We will take

$$q_x(t_1, t_2) = \sigma_0^2 e^{-r|t_1 - t_2|}$$

From (11) it follows that, in this case,

$$\sigma_\Delta^2(t) \leq \sigma_0^2 \frac{3}{8} \sqrt{\frac{2}{\pi}} t^3 - \frac{7}{8^2} t^2 - \frac{2}{8^3} t e^{-rt} + \frac{2}{8^4} (1 - e^{-rt}) \quad (12)$$

We will assume for definiteness that $r = 1 \text{ sec}^{-1}$ and will consider that in constructing a scheme for optimum unbiased filtration which converts $Z_x(t)$ into $Z_x^*(t)$ and in computing $\sigma_{Z_x^*}^2$ -- the dispersion of noise on the output of this scheme -- $\Delta(t)$ can be neglected ($Z_{1x}(t) \approx Z_{1x}^*(t)$), if

$$\sigma_\Delta^2(t) < 0.1 \sigma_0^2 \quad (13)$$

From (12) it follows that (13) is satisfied when $t < 1900 \text{ sec}$. In this case the pulse transfer function of the optimum scheme is easily found and

$$\sigma_{Z_x^*}^2 = \sigma_0^2 \frac{24 + 24\gamma T + 8\gamma^2 T^2}{24 + 24\gamma T + 8\gamma^2 T^2 + \gamma^3 T^3} \quad (14)$$

where T is the value of the memory of the optimum scheme and $\sigma_{Z_x^*}^2$ is the dispersion of noise contained in $Z_x^*(t)$.

From (12) and (14) it follows that the addition to the output of the optimum scheme of an integral term in accordance with formula (10) in practice does not increase the random error, if $t \leq T \leq 500$ sec. Therefore the memory of the optimum scheme should be determined by the conditions

$$T = t \quad \text{for } t \leq 500 \text{ sec.},$$

$$T = 500 \text{ sec for } t \geq 500 \text{ sec.}$$

In this case formula (14) can be used to determine the dispersion of random errors in the smoothed coordinates of satellites.

Beginning with the moment $t = 500$ sec the lower limit of the integral in the right-hand members of formulas (9), (10) should be replaced by $t-500$.

It can be shown analogously that this same selection of T makes it also possible to neglect $\Delta(t)$ in the determination of σ_x . Then it is easy to find the pulse transfer function of the scheme for optimum differentiation, and σ_v^2 - dispersion of random errors in determining the velocity projections of the satellite -- is described by the relation

$$\sigma_v^2 = \sigma_0^2 24 T^2 \frac{2 + \frac{t}{T}}{24 \delta T + 24 \delta^2 T^2 + 8 \delta^3 T^3 + 4 \delta^4 T^4} \quad (15)$$

Taking for an example $\sigma_0 = 1 \text{ km}$, we obtain from (14) and (15) that in 200 sec after the beginning of receipt of information on the satellite, its coordinates and the projections of its velocity can be found without dynamic errors and with random errors characterized by the values $\sigma_x = 200 \text{ m}$, $\sigma_v = 1.7 \text{ m/sec}$.

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